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1992 J. Phys. A: Math. Gen. 25 1399

(http://iopscience.iop.org/0305-4470/25/5/038)

J. Phys. A: Math. Gen. 25 (1992) 1399-1403. Printed in the UK

ADDENDUM

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Received 30 August 1991

Abstract. The Aharonov–Bohm chaotic billiards have been introduced by Berry and Robnik as planar domains D containing a single line of magnetic flux, with a charged particle moving in D. Switching on the magnetic flux breaks the time reversal symmetry for the quantal particle, whilst the classical billiard dynamics is preserved, so that the geometry of classical trajectories is unchanged. The boundary of D is such that the classical dynamics is chaotic (ergodic). We present improved statistics of energy levels for the Aharonov–Bohm chaotic billiards, showing that the agreement with the prediction of the Gaussian unitary ensemble of the random matrix theory is excellent.

This paper can be considered as an addendum to the paper by Berry and Robnik (1986), hereafter referred to as I, in which the chaotic Aharonov–Bohm billiards have been introduced and studied. The purpose here is to revise and to significantly improve the statistics of energy levels for the case of the broken time reversal symmetry. It will be shown that the agreement with the prediction of the Gaussian unitary ensemble of the random matrix theory is really excellent.

The system under study is defined as a point particle with charge q and mass m moving freely inside a planar domain D with hard walls (implying a specular reflection at each bounce on the boundary ∂D of the billiard table D) and a single magnetic flux line Φ placed at the origin of the plane $\boldsymbol{r}=(x,y)$. The defining boundary is such that the classical dynamics—which is unaffected by the magnetic flux except for the trajectories crossing the origin, which however have measure zero—is ergodic or more chaotic. The vector potential $\boldsymbol{A}(\boldsymbol{r})$ satisfies the equation

$$\nabla \wedge \boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{n}\Phi\delta(\boldsymbol{r}) \tag{1}$$

where $\boldsymbol{n}$ is a unit vector normal to the plane D. The energy levels are determined by the Schrödinger equation

$$(1/2m)(-\mathrm{i}\hbar\nabla - q\boldsymbol{A}(\boldsymbol{r}))^2\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}) \tag{2}$$

where the eigenfunction $\psi(\boldsymbol{r})$ must be single valued and vanishes on the boundary, i.e. $\psi=0$ on ∂D.

The quantum spectrum is controlled by the *quantum flux parameter*, defined as

$$\alpha = q\Phi/h \tag{3}$$

which can be easily seen by eliminating the vector potential using the Dirac substitution

$$\psi(\boldsymbol{r})=\chi(\boldsymbol{r})\exp\left(\frac{\mathrm{i}q}{\hbar}\int_{\boldsymbol{r}_0}^{\boldsymbol{r}}\boldsymbol{A}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'\right) \tag{4}$$

where $\boldsymbol{r}_0$ is an arbitrary reference point. The Schrödinger equation in polar coordinates now reads

$$(-\hbar^2/2m)\nabla^2\chi(r,\theta)=E\chi(r,\theta) \tag{5}$$

with $\chi=0$ on ∂D. At first glance the magnetic flux has been eliminated completely, which is of course a false impression, because we have to require that the eigenfunction $\psi(r,\theta)$ is single valued, and therefore χ must acquire a phase factor

$$\exp\left(-\frac{\mathrm{i}q}{\hbar}\oint\boldsymbol{A}\,\mathrm{d}\boldsymbol{r}\right)=\exp\left(-\frac{\mathrm{i}q}{\hbar}\Phi\right)=\exp(-2\pi\mathrm{i}\alpha) \tag{6}$$

for the positive circuit around the flux line. Therefore we must require

$$\chi(r,\theta+2\pi)=\exp(-2\pi\mathrm{i}\alpha)\chi(r,\theta). \tag{7}$$

Thus we have the real Schrödinger equation (5) with real boundary condition, but with a complex continuation condition (7), so that the eigenvalue problem is in fact complex. Therefore we predict the Gaussian unitary ensemble statistics of the energy spectrum whenever the continuation condition is complex, and Gaussian orthogonal ensemble statistics in the exceptional cases when the continuation condition (7) is real. The latter happens whenever α is integer or a half-integer. In each of these exceptional cases the system has a non-trivial real representation (it has an anti-unitary symmetry; see Berry and Robnik (1986) and Robnik (1986)), and therefore the Gaussian orthogonal ensemble level statistics applies. It is trivial to see that the spectrum $E_j(\alpha)$ has the symmetries

$$E_j(\alpha+1)=E_j(\alpha)\qquad\text{and}\qquad E_j(-\alpha)=E_j(\alpha). \tag{8}$$

Therefore it is sufficient to study the spectrum of the Aharonov–Bohm billiard only on the interval $0\leqslant\alpha\leqslant\frac{1}{2}$. It is also clear that the case $\alpha=\frac{1}{4}$ is the furthest away from the two real cases $\alpha=0$ and $\alpha=\frac{1}{2}$, i.e. the breaking of the time reversal symmetry (for finite spectral stretches) is the strongest for $\alpha=\frac{1}{4}$. This is the value of the quantum flux parameter that we have chosen to demonstrate the GUE level statistics.

The plane billiards that we choose are described by the boundary curve in the complex w-plane which is a conformal map of the unit circle in the complex z-plane. Thus we choose the simplest but rich enough generic family given by

$$w=z+Bz^2+C\,\mathrm{e}^{\mathrm{i}\phi}z^3 \tag{9}$$

where $z=x+\mathrm{i}y$, $w=u+\mathrm{i}v$, while B and C are real parameters and ϕ is a real angle chosen in such a manner as to warrant the absence of all geometric symmetries such as reflection symmetry. A set of the parameter values B, C, ϕ determines thus a given bounding curve ∂D in the w-plane, which is the image of the unit circle $|z|=1$ of the z-plane.

We have chosen three different sets of parameters defining three different billiard boundaries, as shown in table 1. Each of them has been carefully checked to be numerically ergodic by investigating the Poincare map on the surface of section. As an example, in figure 1 we show the shape of the Africa billiard (number 1 in table 1). All billiards are non-convex shapes. Actually, the non-convexity of the boundaries

Table 1. List of the defining parameters B, C, ϕ for three billiards, including the area $\mathcal{A}$ and perimeter $\mathcal{L}$. The billiard number 1 is the so-called Africa billiard, introduced and extensively studied in Berry and Robnik (1986).

Number	B	C	ϕ	$\mathcal{A}$	$\mathcal{L}$
1	0.2	0.2	$\pi/3$	3.7699	7.1012
2	0.1	0.2	$\pi/3$	3.5814	6.9218
3	0.1	0.3	$\pi/3$	4.0526	7.7082

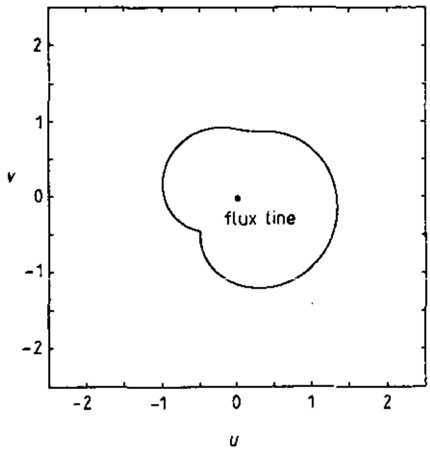


Figure 1. Example: the shape of the Africa billiard (number 1 from table 1). Each billiard boundary is the conformal image of the unit disc $|z| = 1$ of the z-plane according to the equation (1), with the parameters given in table 1.

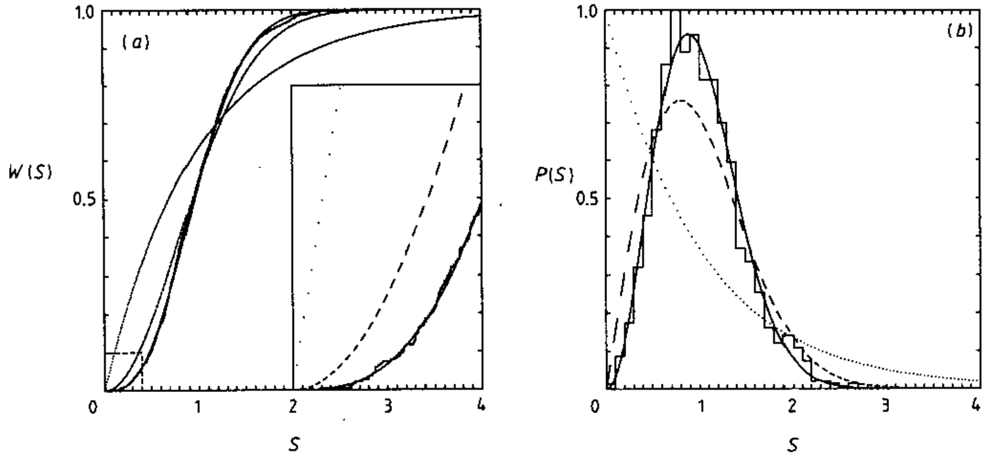
is sufficient to warrant chaotic nature of the billiards, since it breaks all Lazutkin-type caustics in the configuration space and the associated invariant curves in the phase space. This is intuitively obvious, but there is also a theorem by Mather (1982) supporting this view. Some small islands of stability in the vicinity of stable periodic orbits could in principle be present still, but our numerical evidence does indicate that the systems are all ergodic for all practical purposes. Therefore the islands of stability must be very small indeed, if they exist at all.

If there are no singularities of the boundaries then the boundary is analytic and the conformal mapping of the unit disc in the z-plane onto the billiard in the w-plane is one-to-one, i.e. it is bijective. This makes it possible to solve the eigenvalue problem in the z-plane, where we have the basis of the Bessel functions, instead of solving it in the w-plane, where no basis is obvious. This is the main idea of the conformal transformation technique which is described in I and in Robnik (1984, 1991).

Using this method I have calculated $M = 2000$ energy eigenvalues E for each of the three billiards defined in table 1, spending about 30 CPU hours on the VAX8800 computer for each case. Special attention has been paid to the various checks of the accuracy of the numerical diagonalization procedure. It is found that for $M = 2000$ the first 550 levels are more accurate than 1% of the mean energy level spacing, and thus suitable for the significant statistical analysis. The Weyl type formula has been used firstly to check for gross errors, and secondly for the unfolding procedure (see

Bohigas and Giannoni 1984, Robnik 1991). Finally, the lowest fifty levels have been discarded, since they are considered to be far from the semiclassical regime and could spoil the level statistics, so that we have analysed the composite of 3 times $500 = 1500$ levels. The results are excellent as is seen in figure 2(a–c). In each of these figures we show three theoretical curves, namely Poissonian (characteristic of integrable systems), GOE-curve and the theoretically expected GUE-curve. The cumulative spacings $W(S) = \int_0^S P(x)\,dx$, where $P(S)$ is the level spacings distribution, closely follow the theoretical curve of GUE (figure 2(a)), and so does the histogram in figure 2(b). The Δ-statistics shown in figure 2(c) is consistent with the theoretical curve of GUE. The saturation seen in figure 2(c) can be fully understood in terms of the semiclassical theory of spectral rigidity by Berry (1985): the saturation should start at $L_{max} \approx 20$, and the saturation value should be $\Delta_\infty \approx 0.2$. Finally the χ^2 test has been performed for the cumulative spacings distribution $W(S)$, with the result that the confidence level is 80%.

The case of the Aharonov–Bohm billiards with zero flux, $\alpha = 0$, has been recently revised and the GOE-type behaviour confirmed (Robnik 1991), thereby yielding a new firm support for the conjecture by Bohigas, Giannoni and Schmit (1984), that classically ergodic (or more chaotic) systems should display the GOE-type statistics of energy levels if there is time reversal symmetry or some other anti-unitary symmetry (Robnik



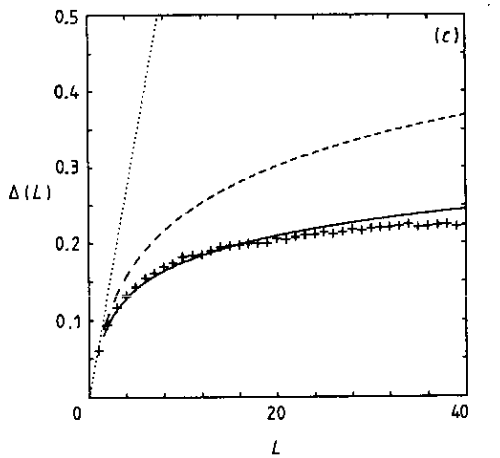


Figure 2. The results for the three billiards of table 1. In each figure there are three theoretical curves: the Poissonian (dotted), the GUE curve (full) and the GOE curve (dashed). (a) The cumulative energy level spacings distribution $W(S) = \int_0^S dt\,P(t)$; (b) energy level spacings histogram with 40 bins; (c) $\Delta(L)$ statistics as a function of L for $0 \leq L \leq 40$; the saturation occurs at $L_{max} \approx 20$.

1986), and should be GUE in cases of broken anti-unitary symmetry. In my view the results of the present work do provide a strong additional evidence for the validity of the two universality classes of spectral statistics for classically ergodic systems.

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